AD-A096 094 PURDUE UNIV LAFAYETTE IN DEPT OF STATISTICS F/G 12/1
ON GAMMA-MINIMAX, MINIMAX, AND BAYES PROCEDURES FOR SELECTING P--ETC(U)
APR 80 S S GUPTA, P HSIAO N00014-75-C-0455
UNCLASSIFIED MMS-80-9 NL

1 OF 1
AD A
096094

END
DATE
FILMED
4-81
DTIC

PURDUE UNIVERSITY

DEPARTMENT OF STATISTICS

DIVISION OF MATHEMATICAL SCIENCES

(1)

6 ON Γ-MINIMAX, MINIMAX, AND BAYES PROCEDURES FOR SELECTING POPULATIONS CLOSE TO A CONTROL*

by

10 Shanti S. Gupta [redacted] Ping Hsiao
Purdue University and University of Michigan

Department of Statistics
Division of Mathematical Sciences
7 Mimeograph Series 2 80-9

11 Apr[redacted]80 12

*This research was supported by the Office of Naval Research contract N00014-75-C-0455 at Purdue University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Gamma

ON Γ-MINIMAX, MINIMAX, AND BAYES PROCEDURES FOR SELECTING POPULATIONS CLOSE TO A CONTROL

by

Shanti S. Gupta and Ping Hsiao
Purdue University and University of Michigan

ABSTRACT

Pi sub ∅ Pi sub 1 Pi sub k

Let $\Pi_0, \Pi_1, \ldots, \Pi_k$ be (k+1) normally distributed populations and let Π_0 be a control population. Our goal is to select those populations which are sufficiently close to the control in terms of the (unknown) means of the populations. A zero-one type loss function is defined. Γ-minimax rules, Bayes rules and minimax rules are derived for this problem and compared. Some optimal properties of Γ-minimax rules are shown; also Γ-minimax rules are derived for distributions other than the normal.

gamma

Accession For

A

ON Γ-MINIMAX, MINIMAX, AND BAYES PROCEDURES FOR SELECTING POPULATIONS CLOSE TO A CONTROL*

by

Shanti S. Gupta and Ping Hsiao
Purdue University and University of Michigan

1. Introduction and summary

Problems of selecting populations close to a control arise frequently in industrial production, in situations such as for matching parts. Assume that we have (k+1) populations and one of them is the control or standard population. Our goal is to select those populations which are sufficiently close to the control. Many authors have considered problems of comparing populations with a control under different types of formulations. Paulson (1952), Bechhofer and Turnbull (1974) discussed problems of selecting the best population if the best population is better than the control. Dunnett (1955), Gupta and Sobel (1958) considered problems of selecting a subset containing all populations better than the control. Lehmann (1961), Tong (1969), Randles and Hollander (1971) dealt with problems of selecting populations better than control. For problems of classifying a set of populations into three groups which are 'superior', 'inferior' and 'equivalent' to a control, see Kim (1979) and Gupta and Kim (1980) and related references therein. However, not much work has been done for the problem of selecting populations close to a control.

*This research was supported by the Office of Naval Research contract N00014-75-C-0455 at Purdue University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Singh (1977) and Gupta and Singh (1979) considered this problem and derived Bayes (and empirical Bayes) rules for various distributions. In this paper, Γ-minimax rules for selecting populations close to a control are derived, and these are compared with minimax rules and Bayes rules for robustness against the prior information. In Section 2, definitions and notations are introduced and a decision theoretic formulation of the problem is given. Results in Section 3 and Section 4, deal with the cases when all populations are assumed to be normally distributed. Γ-minimax decision rules are derived when the control parameter θ_0 is known, and restricted Γ-minimax rules are derived when θ_0 is unknown. In Section 5, some optimal properties of Γ-minimax rules are found. In Section 6, results of Section 3 are generalized and Γ-minimax rules are derived for distributions other than the normal. Γ-minimax rules for selecting binomial populations with large entropy are also discussed. In Section 7, θ_0 is treated as an unknown. Bayes rules are found under the assumptions that θ_i has a normal prior distribution with mean α_i and variance β_i^2, $i = 0, 1, \ldots, k$, which are assumed known. Minimax rules are also derived in this section. And Bayes rules, Γ-minimax rules and minimax rules are compared in Section 8 in terms of the Bayes risk, the maximum risk over Γ and the maximum risk over all the possible prior distributions.

2. Notations and formulation of the problem.

Let $\Pi_0, \Pi_1, \ldots, \Pi_k$ be $(k+1)$ independent normal populations with means $\theta_0, \theta_1, \ldots, \theta_k$ and known variances $\sigma_0^2, \sigma_1^2, \ldots, \sigma_k^2$, respectively. Assume that Π_0 's the control population, with mean θ_0 which may be known or unknown. For $\Pi_1,\ldots,\Pi_k$, the treatment populations, $\theta_1,\ldots,\theta_k$ are all assumed to be unknown. When θ_0 is unknown, let $\underset{\sim}{\theta} = (\theta_0,\theta_1,\ldots,\theta_k)$ and $\underset{\sim}{X} = (X_0,X_1,\ldots,X_k)$ where X_i is an observation from Π_i, $i = 0,1,\ldots,k$. When θ_0 is known, no observation from Π_0 is taken, and θ_0, X_0 are deleted from $\underset{\sim}{\theta}$ and $\underset{\sim}{X}$, respectively. When there is no confusion, $\underset{\sim}{\theta}$ and $\underset{\sim}{X}$ are used to represent either case. Let Θ be the parameter space and χ be the sample space. For $i = 1,2,\ldots,k$, define $G_i = \{\underset{\sim}{\theta}\in\Theta \,\big|\, |\theta_i-\theta_0| \le \Delta\}$ and $B_i = \{\underset{\sim}{\theta}\in\Theta \,\big|\, |\theta_i-\theta_0| \ge \Delta+\varepsilon\}$ where Δ and ε are given positive constants. Π_i is said to be good (or acceptable) if $\underset{\sim}{\theta}\in G_i$ and bad (not acceptable) if $\underset{\sim}{\theta}\in B_i$. We consider decision rules of the form $\underset{\sim}{\delta}(x) = (\delta_1(\underset{\sim}{x}),\ldots,\delta_k(\underset{\sim}{x}))$, where $\delta_i(\underset{\sim}{x})$ denotes the conditional probability of selecting Π_i as a good population given $\underset{\sim}{X}=\underset{\sim}{x}$. The objective is to select all the good populations while rejecting all the bad ones. Let L_1 be the loss incurred when we fail to select a good population and L_2 the loss for each bad population selected. The the loss function is defined by

$$L(\underset{\sim}{\theta},\underset{\sim}{\delta}) = \sum_{i=1}^{k} L_1(1-\delta_i)I_{G_i}(\underset{\sim}{\theta})+L_2\delta_i I_{B_i}(\underset{\sim}{\theta}) = \sum_{i=1}^{k} L^{(i)}(\underset{\sim}{\theta},\delta_i). \quad (2.1)$$

Where I_A denotes the indicator function of A. We assume that the partial information available is of the form: Π_i has probability λ_i to be good and probability λ_i' to be bad. Let $\underset{\sim}{\lambda} = (\lambda_1,\ldots,\lambda_k)$ and $\underset{\sim}{\lambda}' = (\lambda_1',\ldots,\lambda_k')$. We define $\mathscr{G}^* = \{\tau \,|\, \tau$ is a prior distribution on $\Theta\}$, and $\Gamma = \Gamma(\underset{\sim}{\lambda},\underset{\sim}{\lambda}') = \{\tau\in\Theta^* \,|\, P_\tau(G_i) = \lambda_i,\ P_\tau(B_i) = \lambda_i'$, for $i = 1,\ldots,k\}$, where $0 \le \lambda_i + \lambda_i' \le 1$ and $P_\tau(A) = \int_A d\tau(\underset{\sim}{\theta})$. Then, $\Gamma(\underset{\sim}{\lambda},\underset{\sim}{\lambda}')$ denotes the class of all the prior distributions which summarizes our information about $\underset{\sim}{\theta}$. We restrict our investigation to this class. Let $R(\underset{\sim}{\theta},\underset{\sim}{\delta}) = E_{\theta}[L(\underset{\sim}{\theta},\underset{\sim}{\delta}(\underset{\sim}{X}))]$ and $r(\tau,\underset{\sim}{\delta}) = E_{\tau}[R(\underset{\sim}{\theta},\underset{\sim}{\delta})]$. In this framework, an ith component problem is concerned with the selection or rejection of Π_i. Then $R^{(i)}(\underset{\sim}{\theta},\delta_i) = E_{\theta}[L^{(i)}(\underset{\sim}{\theta},\delta_i)]$ and $r^{(i)}(\tau,\delta_i) = E_{\tau}[R^{(i)}(\underset{\sim}{\theta},\delta_i)]$ denote the risk function and the Bayes risk function of the ith component problem, respectively. It is found that

$$R(\underset{\sim}{\theta},\underset{\sim}{\delta}) = \sum_{i=1}^{k} R^{(i)}(\underset{\sim}{\theta},\delta_i) \quad \text{and} \quad r(\tau,\underset{\sim}{\delta}) = \sum_{i=1}^{k} r^{(i)}(\tau,\delta_i).$$

A rule $\underset{\sim}{\delta}^*$ is called a Γ-minimax rule in D if

$$\sup_{\tau\in\Gamma} r(\tau,\underset{\sim}{\delta}^*) = \inf_{\underset{\sim}{\delta}\in D} \sup_{\tau\in\Gamma} r(\tau,\underset{\sim}{\delta})$$

where D is a class of decision rules.

3. Derivation of a Γ-minimax rule when θ_0 is known.

In this section, θ_0 is assumed to be known. We define $G_{i1} = \{\underset{\sim}{\theta}\in G_i | \theta_i=\theta_0+\Delta\}$, $G_{i2} = \{\underset{\sim}{\theta}\in G_i | \theta_i=\theta_0-\Delta\}$, $B_{i1} = \{\underset{\sim}{\theta}\in B_i | \theta_i=\theta_0+\Delta+\epsilon\}$ and $B_{i2} = \{\underset{\sim}{\theta}\in B_i | \theta_i=\theta_0-\Delta-\epsilon\}$. Let $\delta_i(\underset{\sim}{x}) = \delta_i(x_i)$ be an ith component decision rule and let $g_i(\theta_i) = E_{\theta_i}[\delta_i(X_i)]$, then we have

<u>Lemma 3.1.</u> For any fixed i, if $\inf_{\underset{\sim}{\theta}\in G_i} g_i(\theta_i) = g_i(\theta_0+\Delta) = g_i(\theta_0-\Delta)$

and
$$\sup_{\underset{\sim}{\theta}\in B_i} g_i(\theta_i) = g_i(\theta_0+\Delta+\epsilon)=g_i(\theta_0-\Delta-\epsilon), \tag{3.1}$$

then

$$\sup_{\tau\in\Gamma} r^{(i)}(\tau,\delta_i)=r^{(i)}(\tau_0,\delta_i) \text{ for all } \tau_0\in\Gamma_0(i),$$

where $\Gamma_0(i) = \{\tau\in\Gamma | P_{\tau}(G_{i1})+P_{\tau}(G_{i2})=\lambda_i, P_{\tau}(B_{i1})+P_{\tau}(B_{i2})=\lambda_i'\}$.

Proof: $r^{(i)}(\tau,\delta_i) = \int_{G_i} E_{\underset{\sim}{\theta}}[L_1(1-\delta_i(\underset{\sim}{X}))]d\tau(\underset{\sim}{\theta})$

$$+ \int_{B_i} E_{\underset{\sim}{\theta}}[L_2\delta_i(\underset{\sim}{x})]d\tau(\underset{\sim}{\theta})$$

$$\leq L_1\lambda_i - L_1\lambda_i \inf_{\underset{\sim}{\theta}\in G_i} g_i(\theta_i) + L_2\lambda_i' \sup_{\underset{\sim}{\theta}\in B_i} g_i(\theta_i)$$

$$= L_1\lambda_i - L_1\{P_{\tau_0}(G_{i1})g_i(\theta_0+\Delta) + P_{\tau_0}(G_{i2})g_i(\theta_0-\Delta)\}$$

$$+ L_2\{P_{\tau_0}(B_{i1})g_i(\theta_0+\Delta+\epsilon) + P_{\tau_0}(B_{i2})g_i(\theta_0-\Delta-\epsilon)\}$$

$$= r^{(i)}(\tau_0,\delta_i) \text{ for all } \tau_0 \in \Gamma_0(i).$$

This completes the proof.

Theorem 3.1. If there exists a $\tau^* \in \bigcap_{i=1}^{k} \Gamma_0(i)$ such that $\delta_i^*(\underset{\sim}{x}) = \delta_i^*(x_i)$ is a Bayes rule wrt τ^* for the ith component problem and if (3.1) is satisfied for $g_i(\theta_i) = E_{\theta_i}[\delta_i^*(X_i)]$ for all $i=1,2,\ldots,k$, then $\delta^* = (\delta_1^*,\ldots,\delta_k^*)$ is a Γ-minimax rule.

Proof: $\sup_{\tau\in\Gamma} r(\tau,\underset{\sim}{\delta}^*) \leq \sum_{i=1}^{k} \sup_{\tau\in\Gamma} r^{(i)}(\tau,\delta_i^*)$

$$= \sum_{i=1}^{k} r^{(i)}(\tau^*,\delta_i^*) \qquad \text{by Lemma 3.1}$$

$$\leq \sum_{i=1}^{k} r^{(i)}(\tau^*,\delta_i) = r(\tau^*,\delta) \leq \sup_{\tau\in\Gamma} r(\tau,\delta)$$

for all δ. This completes the proof.

Lemma 3.2. Let the pdf $f(x|\theta)$ of X be TP_3 (Totally Positive of order 3). If $g(\theta) = E_\theta[I_{(a,b)}(X)]$, and for some θ_0, $g(\theta+\theta_0) = g(\theta-\theta_0)$, then g is increasing for $\theta < \theta_0$ and hence decreasing for $\theta > \theta_0$.

Proof: Let $h_c(x) = I_{(a,b)}(x)-c$ for $c \in (0,1)$, then $g_c(\theta) = g(\theta)-c$ where $g_c(\theta) = E_\theta[h_c(X)]$. Let $S(h_c)$ denote the number of sign changes of the function h_c, then $S(h_c) = 2$. Now by Variation Diminishing Property (VDP)(Karlin (1968), see p. 21) it is seen that $S(g_c) \leq 2$ for all $c \in (0,1)$. If g is not increasing for $\theta < \theta_0$, then there exist $\theta_1 < \theta_2 < \theta_0$ and $g(\theta_1) > g(\theta_2)$. Let $\theta_1' = 2\theta_0-\theta_1$ and $\theta_2' = 2\theta_0-\theta_2$, then $g(\theta_1') > g(\theta_2')$. We find that $S(g_{c_0}) \geq 2$ for $c_0 = 1/2[g(\theta_1)+g(\theta_2)]$, so $S(g_{c_0}) = 2$. But g_{c_0} does not change signs in the same way as h_{c_0} does which contradicts VDP. This completes the proof.

Now let

$$g_{\sigma_i}(x) = \frac{1}{\sigma_i}\left[\phi\left(\frac{x+\Delta}{\sigma_i}\right)+\phi\left(\frac{x-\Delta}{\sigma_i}\right)\right]$$

$$f_{\sigma_i}(x) = \frac{1}{\sigma_i}\left[\phi\left(\frac{x+\Delta+\varepsilon}{\sigma_i}\right)+\phi\left(\frac{x-\Delta-\varepsilon}{\sigma_i}\right)\right]. \tag{3.2}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Then we have

<u>Theorem 3.2</u>. If $\delta_i^*(\underset{\sim}{x}) = \delta_i^*(x_i) = I_{[-t_i,t_i]}(x_i-\theta_0)$ and $t_i > 0$ satisfies

$$L_2\lambda_i' f_{\sigma_i}(t_i) = L_1\lambda_i g_{\sigma_i}(t_i) \text{ for } i=1,\ldots,k, \tag{3.3}$$

then $\delta^* = (\delta_1^*,\ldots,\delta_k^*)$ is a Γ-minimax rule.

Proof: Let $\tau^* \in \Gamma$ be a prior distribution on Θ such that $\theta_1,\theta_2,\ldots,\theta_k$ are independent under τ^*, and $P_{\tau^*}(\theta_i=\theta_0+\Delta+\frac{\varepsilon}{2})=1-\lambda_i-\lambda_i'$, $P_{\tau^*}(G_{i1})$ $P_{\tau^*}(G_{i2}) = \frac{\lambda_i}{2}$, $P_{\tau^*}(B_{i1})=P_{\tau^*}(B_{i2}) = \frac{\lambda_i'}{2}$ for $i=1,2,\ldots,k$.

Let $\quad f(\underset{\sim}{x}|\underset{\sim}{\theta}) = \prod_{i=1}^{k} \frac{1}{\sigma_i}\phi\left(\frac{x_i-\theta_i}{\sigma_i}\right)$, then we have

$$r^{(i)}(\tau^*,\delta_i) = \int_{\chi} L_1(1-\delta_i(x)) \sum_{\underset{\sim}{\theta}\in G_{i1}\cup G_{i2}} f(\underset{\sim}{x}|\underset{\sim}{\theta})P_{\tau^*}(\underset{\sim}{\theta})$$

$$+ L_2\delta_i(\underset{\sim}{x}) \sum_{\underset{\sim}{\theta}\in B_{i1}\cup B_{i2}} f(\underset{\sim}{x}|\underset{\sim}{\theta})P_{\tau^*}(\underset{\sim}{\theta})d\underset{\sim}{x}.$$

If we let $C(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k) = \dfrac{\sum_{\underset{\sim}{\theta}\in G_{i1}} f(\underset{\sim}{x}|\underset{\sim}{\theta})P_{\tau^*}(\underset{\sim}{\theta})}{\frac{\lambda_i}{2}\frac{1}{\sigma_i}\phi(\frac{x_i-\theta_0-\Delta}{\sigma_i})}$

$$= \frac{\sum_{\underset{\sim}{\theta}\in G_{i2}} f(\underset{\sim}{x}|\underset{\sim}{\theta})P_{\tau^*}(\underset{\sim}{\theta})}{\frac{\lambda_i}{2}\frac{1}{\sigma_i}\phi(\frac{x_i-\theta_0+\Delta}{\sigma_i})} = \frac{\sum_{\underset{\sim}{\theta}\in B_{i1}} f(\underset{\sim}{x}|\underset{\sim}{\theta})P_{\tau^*}(\underset{\sim}{\theta})}{\frac{\lambda_i'}{2}\frac{1}{\sigma_i}\phi(\frac{x_i-\theta_0-\Delta-\varepsilon}{\sigma_i})}$$

$$= \frac{\sum_{\underset{\sim}{\theta}\in B_{i2}} f(\underset{\sim}{x}|\underset{\sim}{\theta})P_{\tau^*}(\underset{\sim}{\theta})}{\frac{\lambda_i'}{2}\frac{1}{\sigma_i}\phi(\frac{x_i-\theta_0+\Delta+\varepsilon}{\sigma_i})}, \text{ then}$$

$$r^{(i)}(\tau^*,\delta_i) = 1/2\int_{\chi} L_1\lambda_i g_{\sigma_i}(x_i-\theta_0)C(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)dx$$

$$+ 1/2\int_{\chi}\delta_i(x)[L_2\lambda_i' f_{\sigma_i}(x_i-\theta_0)-L_1\lambda_i g_{\sigma_i}(x_i-\theta_0)]c(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)dx.$$

Hence the Bayes rule for the ith component problem wrt τ^* is

$$\delta_i^*(x) = \delta_i^*(x_i) = \begin{cases} 1 & \text{if } L_1\lambda_i g_{\sigma_i}(x_i-\theta_0) \geq L_2\lambda_i' f_{\sigma_i}(x_i-\theta_0) \\ 0 & \text{if } \qquad < \end{cases}$$

Let $h_i(x) = \dfrac{L_2\lambda_i' f_{\sigma_i}(x)}{L_1\lambda_i g_{\sigma_i}(x)} = k_i \dfrac{\cosh[\frac{x}{\sigma_i^2}(\Delta+\varepsilon)]}{\cosh[\frac{x}{\sigma_i^2}\Delta]}$

where $k_i = \dfrac{L_2\lambda_i'}{L_1\lambda_i}\exp[-\dfrac{1}{2\sigma_i^2}(2\Delta+\varepsilon)\varepsilon]$, then $h_i(x)=h_i(-x)$ and $h_i(x)$ is increasing for $x > 0$, hence $h_i(x) \leq 1$ if and only if $|x| \leq t_i$, where $t_i > 0$ satisfies $h_i(t_i) = 1$. So, $\delta_i^*(x_i)=I_{[-t_i,t_i]}(x_i-\theta_0)$.

Now, if $g_i(\theta_i) = E_{\theta_i}[\delta_i^*(X_i)]$, we find $g_i(\theta_i+\theta_0)=g_i(\theta_0-\theta_i)$. Also, $X_i \sim N(\theta_i,\sigma_i^2)$, so the pdf of X_i is TP, hence TP_3 from Karlin (1968) (see p. 18). Now, by Lemma 3.2, (3.1) is satisfied, then Theorem 3.1 shows that $\underset{\sim}{\delta}^*$ is a Γ-minimax rule. This finishes the proof.

Let $\underset{\sim}{\lambda}/\underset{\sim}{\lambda}'$ be defined as $(\frac{\lambda_1}{\lambda_1'},\ldots,\frac{\lambda_k}{\lambda_k'})$. If $\Gamma(\gamma)=\{\tau\in\Theta^*|P_\tau(G_i)/P_\tau(B_i)=\gamma_i$ for $i=1,2,\ldots,k\}$ where $\underset{\sim}{\gamma}=(\gamma_1,\gamma_2,\ldots,\gamma_k)$, then we have $\Gamma(\gamma)= \bigcup_{\underset{\sim}{\lambda}/\underset{\sim}{\lambda}'=\gamma} \Gamma(\underset{\sim}{\lambda},\underset{\sim}{\lambda}')$. Since $\underset{\sim}{\delta}^*$ depends on $\underset{\sim}{\lambda},\underset{\sim}{\lambda}'$ only through $\underset{\sim}{\lambda}/\underset{\sim}{\lambda}'$, we find

$$\sup_{\tau\in(\underset{\sim}{\gamma})} r(\tau,\underset{\sim}{\delta}) = \sup_{\underset{\sim}{\lambda}/\underset{\sim}{\lambda}'=\gamma}\ \sup_{\tau\in\Gamma(\underset{\sim}{\lambda},\underset{\sim}{\lambda}')} r(\tau,\underset{\sim}{\delta})$$

$$\geq \sup_{\underset{\sim}{\lambda}/\underset{\sim}{\lambda}'=\gamma}\ \sup_{\tau\in\Gamma(\underset{\sim}{\lambda},\underset{\sim}{\lambda}')} r(\tau,\underset{\sim}{\delta}^*)$$

$$= \sup_{\tau\in\Gamma(\underset{\sim}{\gamma})} r(\tau,\underset{\sim}{\delta}^*)$$

hence $\underset{\sim}{\delta}^*$ is a Γ-minimax rule for $\Gamma = \Gamma(\underset{\sim}{\gamma})$.

It is possible that (3.3) does not have a solution. In this case, the Γ-minimax rules imply that all populations are bad.

4. A restricted Γ-minimax rule for θ_0 unknown.

When θ_0 is unknown, decision rules are restricted in a subclass D', where $D' = \{\underset{\sim}{\delta}=(\delta_1,\ldots,\delta_k)|\delta_i(x)=\delta_i(x_0,x_i)$ for $i=1,\ldots,k\}$. That such a restriction is needed was first pointed out by Randles and Hollander (14). The following lemma has been used by Miescke (1979). The original idea of this lemma is due to Ferguson (1967) and Lehmann (1959).

Lemma 4.1. Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of prior distributions on Θ, and let δ_{in}^0 be a Bayes rule wrt τ_n for the ith component problem. If

$$\liminf_{n\to\infty} \gamma^{(i)}(\tau_n,\delta_{in}^0) \geq \sup_{\tau\in\Gamma} \gamma^{(i)}(\tau,\delta_i^0)$$

for all $i = 1,2,\ldots,k$, then $\underset{\sim}{\delta}^0 = (\delta_1^0,\delta_2^0,\ldots,\delta_k^0)$ is a Γ-minimax rule.

A prior distribution τ on $\Theta_0 \times \Theta_1 \times\ldots\times \Theta_k$ can be specified by the marginal distribution T on Θ_0 and the conditional distribution ω_{θ_0} on $\Theta_1\times\ldots\times \Theta_k$, given $\theta_0=\theta_0$. We use $\tau=(T,\omega_{\theta_0})$ to denote such prior distributions. Let $\tau_n = (T_n,\omega^*_{\theta_0})$, where T_n is uniformly distributed over the interval $[-n,n]$ and $\theta_1,\theta_2,\ldots,\theta_k$ are conditionally independent under $\omega^*_{\theta_0}$, and

$$P_{\omega^*_{\theta_0}}(B_{i1}|\theta_0) = P_{\omega^*_{\theta_0}}(B_{i2}|\theta_0) = \frac{\lambda_i'}{2}$$

$$P_{\omega^*_{\theta_0}}(G_{i1}|\theta_0) = P_{\omega^*_{\theta_0}}(G_{i2}|\theta_0) = \frac{\lambda_i}{2} \qquad (4.1)$$

$$P_{\omega^*_{\theta_0}}\{\theta|\theta_i=\theta_0+\Delta+\frac{\varepsilon}{2}\} = 1-\lambda_i-\lambda_i' \text{ for } i=1,\ldots,k.$$

Let $\omega^*_{\theta_{0,i}}$ denote the conditional marginal distribution of θ_i under $\omega_{\theta_0^*}$. Then, we have

<u>Theorem 4.1</u>. When θ_0 is unknown, a Γ-minimax rule in D' is given by $\delta^0 = (\delta_1^0,\ldots,\delta_k^0)$, where $\delta_i^0(\underset{\sim}{x}) = I_{[-t_i',t_i']}(x_i-x_0)$ and $t_i' > 0$ satisfies

$$L_1\lambda_i g_{\sigma_i'}(t_i')=L_2\lambda_i' f_{\sigma_i'}(t_i'), \text{ with } \sigma_i'=(\sigma_0^2+\sigma_i^2)^{1/2}, \qquad (4.2)$$

$g_{\sigma_i'}$ and $f_{\sigma_i'}$ are defined in (3.2).

Proof: For τ_n defined above, let $h_\sigma(x) = \frac{1}{\sigma}\phi(\frac{x}{\sigma})$, then

$$r^{(i)}(\tau_n,\delta_i)=\int_{-n}^{n}\{\int_{|\theta_i-\theta_0|\leq\Delta} L_1(1-E_{\theta_0,\theta_i}[\delta_i(X_0,X_i)])d\omega^*_{\theta_{0,i}}(\theta_i)$$

$$+\int_{|\theta_i-\theta_0|\geq\Delta+\varepsilon} L_2E_{\theta_0,\theta_i}[\delta_i(X_0,X_i)]d\omega^*_{\theta_{0,i}}(\theta_i)\}dT_n(\theta_0)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{L_1\lambda_i}{2} \int_{-n}^{n} g_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)\ \frac{1}{2n}\ d\theta_0 dx_i dx_0$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_i(x_0,x_i)\{\frac{L_2\lambda_i'}{2} \int_{-n}^{n} f_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)\ \frac{1}{2n}\ d\theta_0$$

$$- \frac{L_1\lambda_i}{2} \int_{-n}^{n} g_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)\ \frac{1}{2n}\ d\theta_0\}dx_i dx_0.$$

Hence, the Bayes rule wrt τ_n for the ith component problem is

$$\delta_{in}^0(x_i,x_0) = \begin{cases} 1 & \text{if} \quad L_2\lambda_i' \int_{-n}^{n} f_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)d\theta_0 \\ & \quad \le L_1\lambda_i \int_{-n}^{n} g_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)d\theta_0 \\ 0 & \quad > \end{cases}$$

and the Bayes risk of δ_{in}^0 is given by

$$r^{(i)}(\tau_n,\delta_{in}^0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \min\{\frac{L_1\lambda_i}{4n} \int_{-n}^{n} g_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)d\theta_0,$$

$$\frac{L_2\lambda_i'}{4n} \int_{-n}^{n} f_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_i-\theta_0)d\theta_0\}dx_i dx_0.$$

Now consider the change of variables

$$\begin{cases} x_i=ny_i+y_0 \\ x_0=ny_i-y_0 \end{cases}$$ for the outside two integrals, then let $\theta_0=ny_i-\eta_0$ for the inside integral.

Since $\left|\frac{\partial(x_i,x_0)}{\partial(y_i,y_0)}\right| = 2n$ and $h_0(x)=h_0(-x)$, we find

$$r^{(i)}(\tau_n,\delta_{in}^0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \min\{\frac{L_1\lambda_i}{2} \int_{n(y_i-1)}^{n(y_i+1)} g_{\sigma_i}(y_0+\eta_0)h_{\sigma_0}(\eta_0-y_0)d\eta_0,$$

$$\frac{L_2\lambda_i'}{2} \int_{n(y_i-1)}^{n(y_i+1)} f_{\sigma_i}(y_0+\eta_0)h_{\sigma_0}(\eta_0-y_0)d\eta_0\}dy_i dy_0.$$

It is known that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\eta}\,\phi\left(\frac{x-a}{\sigma}\right)\phi\left(\frac{x-b}{\eta}\right)dx = \frac{1}{\sqrt{\sigma^2+\eta^2}}\,\phi\left(\frac{b-a}{\sqrt{\sigma^2+\eta^2}}\right),$$

hence

$$\int_{-\infty}^{\infty} g_{\sigma_i}(\eta_0-a)h_{\sigma_0}(\eta_0-b)d\eta_0 = g_{\sigma_i'}(a-b)$$

and (4.3)

$$\int_{-\infty}^{\infty} f_{\sigma_i}(\eta_0-a)h_{\sigma_0}(\eta_0-b)d\eta_0 = f_{\sigma_i'}(a-b),$$

where

$\sigma_i' = (\sigma_i^2+\sigma_0^2)^{1/2}$. Now by Fatou's lemma and (4.3), we get

$$\liminf_{n\to\infty} r^{(i)}(\tau_n,\delta_{in}^0) \geq \int_{-\infty}^{\infty}\int_{-1}^{1} \min\left\{\frac{L_1\lambda_i}{2}\, g_{\sigma_i'}(2y_0),\ \frac{L_2\lambda_i'}{2}\, f_{\sigma_i'}(2y_0)\right\}dy_i\,dy_0 \tag{4.4}$$

$$= \int_{-\infty}^{\infty} \min\left\{\frac{L_1\lambda_i}{2}\, g_{\sigma_i'}(x),\ \frac{L_2\lambda_i'}{2}\, f_{\sigma_i'}(x)\right\}dx.$$

On the other hand, for all $\tau=(T,\omega_{\theta_0})\in\Gamma$

$$\gamma^{(i)}(\tau,\delta_i^0) = \int_{-\infty}^{\infty}\int_{|\theta_i-\theta_0|\leq\Delta} L_1(1-E_{\theta_i-\theta_0}[\delta_i^0(X_i-X_0)])\,d\omega_{\theta_{0,i}}(\theta_i)$$

$$+\int_{|\theta_i-\theta_0|\geq\Delta+\varepsilon} L_2E_{\theta_i-\theta_0}[\delta_i^0(X_i-X_0)]\,d\omega_{\theta_{0,i}}(\theta_i)\,dT(\theta_0)$$

$$\leq L_1\lambda_i[1-\inf_{|\eta_i|\leq\Delta} g_i(\eta_i)]+L_2\lambda_i'\sup_{|\eta_i|\geq\Delta+\varepsilon} g_i(\eta_i),$$

where $\eta_i = \theta_i-\theta_0$, $g_i(\eta_i)=E_{\eta_i}[\delta_i^0(Y_i)]$ and $Y_i=X_i-X_0$.

Since $Y_i\sim N(\eta_i,\sigma_i'^2)$, so as was shown in the proof of Theorem 3.2, we have

$$\sup_{|\eta_i|\geq\Delta+\varepsilon} g_i(\eta_i)=g_i(\Delta+\varepsilon)=g_i(-\Delta-\varepsilon)$$

and (4.5)

$$\inf_{|\eta_i|\leq\Delta} g_i(\eta_i)=g_i(\Delta)=g_i(-\Delta).$$

Thus,

$$r^{(i)}(\tau,\delta_i^0) \le L_1\lambda_i[1-\frac{g_i(\Delta)+g_i(-\Delta)}{2}]+L_2\lambda_i'[\frac{g_i(\Delta+\varepsilon)+g_i(-\Delta-\varepsilon)}{2}]$$

$$= \int_{-\infty}^{\infty}\frac{L_1\lambda_i}{2}[1-\delta_i^0(y_i)]g_{\sigma_i'}(y_i)+\frac{L_2\lambda_i'}{2}\delta_i^0(y_i)f_{\sigma_i'}(y_i)dy_i$$

$$= \int_{-\infty}^{\infty}\min\{\frac{L_1\lambda_i}{2}g_{\sigma_i'}(x), \frac{L_2\lambda_i'}{2}f_{\sigma_i'}(x)\}dx. \tag{4.6}$$

By (4.4), $\sup_{\tau\in\Gamma} r^{(i)}(\tau,\delta_i^0) \le \liminf_{n\to\infty} r(\tau_n,\delta_{in}^0)$

for all $i=1,2,\ldots,k$. Lemma 4.1 now implies that $\underset{\sim}{\delta}^0=(\delta_1^0,\ldots,\delta_k^0)$ is a Γ-minimax rule in D'. This completes the proof.

5. Optimal properties of the Γ-minimax rule.

Suppose that we have n_i independent observations X_{i1}, $X_{i2},\ldots,X_{in_i}$ from Π_i, $i=1,2,\ldots,k$.

Let $\bar{X}_{in_i} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$, then the Γ-minimax rule is of the form

$$\delta_i^*(\bar{x}_{in_i})=I_{[-t_i(n_i),t_i(n_i)]}(\bar{x}_{in_i}-\theta_0) \tag{5.1}$$

where $t_i(n_i)$ is the positive root of the equation

$$h_i(x)=k_i' \cosh\{n_i(\Delta+\varepsilon)x/\sigma_i^2\}/\cosh\{n_i\Delta x/\sigma_i^2\} = 1$$

with $k_i' = L_2\lambda_i'/(L_1\lambda_i)\cdot\exp\{-n_i(2\Delta+\varepsilon)\varepsilon/2\sigma_i^2\}$.

Consider

$$f_i(x)=k_i'\exp\{n_i(\Delta+\varepsilon)x/\sigma_i^2\}/\exp\{n_i\Delta x/\sigma_i^2\}$$

and

$$g_i(x) = \frac{1}{2}k_i'\exp\{n_i(\Delta+\varepsilon)x/\sigma_i^2\}/\exp\{n_i\Delta x/\sigma_i^2\}.$$

Then, $g_i(x) \le h_i(x) \le f_i(x)$, for $x > 0$.

Let $r_i(n_i)$ and $s_i(n_i)$ be the only positive root of $g_i(x)=0$ and $f_i(x)=0$ respectively, then $r_i(n_i) \geq t_i(n_i) \geq s_i(n_i)$.

Now, $$r_i(n_i) = \Delta + \frac{\varepsilon}{2} - \frac{\sigma_i^2 \ell n(L_2\lambda_i'/2L_1\lambda_i)}{n_i\varepsilon}$$

$$s_i(n_i) = \Delta + \frac{\varepsilon}{2} - \frac{\sigma_i^2 \ell n(L_2\lambda_i'/L_1\lambda_i)}{n_i\varepsilon},$$

hence $\lim_{n_i\to\infty} \gamma_i(n_i) = \lim_{n_i\to\infty} t_i(n_i) = \lim_{n_i\to\infty} s_i(n_i) = \Delta + \frac{\varepsilon}{2}$.

Then,

$$\liminf_{n_i\to\infty} \inf_{|\theta_i-\theta_0|\leq\Delta} E_{\theta_i}[\delta_i^*(\bar{X}_{in_i})]$$

$$= \lim_{n_i\to\infty} [\Phi(\frac{t_i(n_i)-\Delta}{\sigma_i\sqrt{n_i}}) - \Phi(\frac{-t_i(n_i)-\Delta}{\sigma_i\sqrt{n_i}})]=1 \tag{5.2}$$

and

$$\lim_{n_i\to\infty} \sup_{|\theta_i-\theta_0|\geq\Delta+\varepsilon} E_{\theta_i}[\delta_i^*(\bar{X}_{in_i})]$$

$$= \lim_{n_i\to\infty} [\Phi(\frac{t_i(n_i)-(\Delta+\varepsilon)}{\sigma_i\sqrt{n_i}}) - \Phi(\frac{-t_i(n_i)-(\Delta+\varepsilon)}{\sigma_i\sqrt{n_i}})] = 0, \tag{5.3}$$

for all $i=1,2,\ldots,k$.

<u>Theorem 5.1</u>. $$\lim_{\min(n_1,\ldots,n_k)\to\infty} \sup_{\tau\in\Gamma} r(\tau,\underset{\sim}{\delta}^*) = 0,$$

where $\delta^* = (\delta_1^*,\ldots,\delta_k^*)$ is the Γ-minimax rule with δ_i^* defined by (5.1), for $i=1,2,\ldots,k$.

Proof: $$\sup_{\tau\in\Gamma} r(\tau,\delta^*) \leq \sum_{i=1}^{k} \sup_{\tau\in\Gamma} r^{(i)}(\tau,\delta_i^*)$$

$$\leq \sum_{i=1}^{n} L_1\lambda_i\left(1-\inf_{|\theta_i-\theta_0|\leq\Delta} E_{\theta_i}[\delta_i^*(\bar{X}_{in_i})]\right)$$

$$+ L_2\lambda_i' \sup_{|\theta_i-\theta_0|\geq\Delta+\varepsilon} E_{\theta_i}[\delta_i^*(\bar{X}_{in_i})].$$

Hence, by (5.2) and (5.3)

$\lim_{\min(n_1,\ldots,n_k)\to\infty} \sup_{\tau\in\Gamma} r(\tau,\underset{\sim}{\delta}^*) = 0.$ This completes the proof.

When θ_0 is unknown, let $\underset{\sim}{\delta}^0=(\delta_1^0,\ldots,\delta_k^0)$ and

$$\delta_i^0(\bar{x}_{in_i},\bar{x}_{0n_0})=I_{[-t_i'(n_i,n_0),t_i'(n_i,n_0)]}(\bar{X}_{in_i}-\bar{X}_{0n_0}),$$

where $t_i'(n_i,n_0)$ is defined in (4.2) with σ_i^2 and σ_0^2 replaced by σ_i^2/n_i and σ_0^2/n_0 respectively. Then $\lim_{\min(n_0,\ldots,n_k)\to\infty} \sup_{\tau\in\Gamma} \gamma(\tau,\underset{\sim}{\delta}^0)=0$ also holds. The proof is thus similar to that of Theorem 5.1 and is omitted.

In deriving the Γ-minimax rule $\underset{\sim}{\delta}^*$, we have proved that $\underset{\sim}{\delta}^*$ is a Bayes rule wrt τ^*. It is easily seen from (3.4) that $\underset{\sim}{\delta}^*$ is the unique Bayes rule wrt τ^*, and hence it is admissible.

<u>Theorem 5.2</u>. When θ_0 is unknown, the Γ-minimax rule $\underset{\sim}{\delta}^0=(\delta_1^0,\ldots,\delta_k^0)$ is admissible in D'.

Proof: Let $\tau_0=(T_0,\omega_{\theta_0}^*)$ be a measure on Θ such that T_0 is Lebesgue measure on Θ_0 and $\omega_{\theta_0}^*$ is defined by (4.1). Then for all $\underset{\sim}{\delta}\in D'$,

$$r(\tau_0,\underset{\sim}{\delta})=\sum_{i=1}^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\lambda_i}{2}L_1(1-\delta_i(x_0,x_i))g_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)$$

$$+\frac{\lambda_i'}{2}L_2\delta_i(x_0,x_i)f_{\sigma_i}(x_i-\theta_0)h_{\sigma_0}(x_0-\theta_0)d\theta_0dx_0dx_i$$

$$=\sum_{i=1}^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\lambda_i}{2}L_1g_{\sigma_i'}(x_i-x_0)dx_0dx_i$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_i(x_i,x_0)\left[\frac{L_2\lambda_i'}{2}f_{\sigma_i'}(x_i-x_0)-\frac{\lambda_i L_1}{2}g_{\sigma_i'}(x_i-x_0)\right]dx_i dx_0$$

by (4.3).

Hence, the generalized Bayes rule is given by $\delta^0=(\delta_1^0,\ldots,\delta_k^0)$ where

$$\delta_i^0(x_i-x_0)=\begin{cases}1 & \text{if } \lambda_i' L_2 f_{\sigma_i'}(x_i-x_0) \le \lambda_i L_1 g_{\sigma_i'}(x_i-x_0)\\ 0 & \qquad\qquad\qquad\qquad\qquad > \end{cases}$$

which is exactly the rule we defined in (4.2). Also, δ^0 is the unique (up to equivalence) generalized Bayes rule wrt τ_0 in D', and $r(\tau_0,\delta^0) < \infty$. Hence δ^0 is admissible in D'. This completes the proof of Theorem 5.2.

6. Relaxing the assumption of normality.

In this section, Π_i's are not limited to be normal populations. Let X_i be an observation from Π_i with pdf $f_i(x|\theta_i)$ and let $A_i = \{x|L_2\lambda_i'[f_i(x|\theta_0+\Delta+\varepsilon)+f_i(x|\theta_0-\Delta-\varepsilon)]$

$$\le L_1\lambda_i[f_i(x|\theta_0+\Delta)+f_i(x|\theta_0-\Delta)]\}\text{ for } i=1,2,\ldots,k. \qquad (6.1)$$

Theorem 6.1. Let $g_i(\theta_i) = E_{\theta_i}[I_{A_i}(X_i)]$ where A_i is defined by (6.1). If $g_i(\theta_i+\theta_0)=g_i(\theta_0-\theta_i)$ and g_i is increasing for $\theta_i > \theta_0$ for all $i=1,2,\ldots,k$, then $\delta=(\delta_1,\ldots,\delta_k)$ is a Γ-minimax rule where $\delta_i(x)=I_{A_i}(x_i)$.

Proof: Let τ^* be defined as in the proof of Theorem 3.2, then the Bayes rule wrt τ^* for the ith component problem is given by

$\delta_i(\underset{\sim}{x}) = I_{A_i}(x_i)$. Now, since g_i is symmetric about θ_0 and $g_i(\theta_i)$ is increasing for $\theta_i > \theta_0$, so

$$\sup_{|\theta_i-\theta_0|\geq\Delta+\varepsilon} g_i(\theta_i)=g_i(\theta_0+\Delta+\varepsilon)=g_i(\theta_0-\Delta-\varepsilon) \text{ and}$$

$$\inf_{|\theta_i-\theta_0|\leq\Delta} g_i(\theta_i)=g_i(\theta_0+\Delta)=g_i(\theta_0-\Delta).$$

Then by Theorem 3.1, we conclude that $\underset{\sim}{\delta}$ is a Γ-minimax rule. As an example of this theorem, we consider the problem of selecting binomial populations with entropy larger than a given constant. For $i=1,2,\ldots,k$, let $\Pi_i \sim b(n_i,\theta_i)$ and $\Psi(\theta_i) = -\theta_i \ell n\theta_i - (1-\theta_i)\ell n(1-\theta_i)$ where n_i is known and θ_i unknown. $\Psi(\theta_i)$ is the entropy associated with Π_i. Define Π_i to be good if $\Psi(\theta_i) \geq \beta+\varepsilon'$ and bad if $\Psi(\theta_i) \leq \beta$. This is equivalent to saying that Π_i is good if $|\theta_i - \frac{1}{2}| \leq \Delta$ and Π_i is bad if $|\theta_i - \frac{1}{2}| \geq \Delta+\varepsilon$ (6.2)

where Δ and ε satisfy $\psi(\frac{1}{2}+\Delta)=\beta+\varepsilon'$, and $\psi(\frac{1}{2}+\Delta+\varepsilon)=\beta$.

Let
$$h_i(x) = \frac{L_2\lambda_i'}{L_1\lambda_i}\;\frac{(\frac{1}{2}+\Delta+\varepsilon)^x(\frac{1}{2}-\Delta-\varepsilon)^{n_i-x}+(\frac{1}{2}+\Delta+\varepsilon)^{n_i-x}(\frac{1}{2}-\Delta-\varepsilon)^x}{(\frac{1}{2}+\Delta)^x(\frac{1}{2}-\Delta)^{n_i-x}+(\frac{1}{2}+\Delta)^{n_i-x}(\frac{1}{2}-\Delta)^x}$$

we find $h_i(\frac{n_i}{2}+x) = h_i(\frac{n_i}{2}-x)$ and

$\frac{h_i(x+1)}{h_i(x)} >, =, < 1$ if $x >, =, < \frac{n_i-1}{2}$. Hence $h_i(x)$ is decreasing for $x < \frac{n_i}{2}$ and increasing for $x > \frac{n_i}{2}$. Now, in view of (6.2), we find $\theta_0 = \frac{1}{2}$, so that

$$A_i = \{x \mid L_2\lambda_i'[f_i(x|\tfrac{1}{2}+\Delta+\varepsilon)+f_i(x|\tfrac{1}{2}-\Delta-\varepsilon)]$$
$$< L_1\lambda_i[f_i(x|\tfrac{1}{2}+\Delta)+f_i(x|\tfrac{1}{2}-\Delta)]\} = \{x \mid h_i(x) < 1\}$$

$= \{x | \frac{n_i}{2} - m_i \le x \le \frac{n_i}{2} + m_i\}$, where $f_i(x|\theta) = \binom{n_i}{x} \theta^x (1-\theta)^{n_i - x}$ and

$\frac{n_i}{2} + m_i$ satisfies $h_i(\frac{n_i}{2} + m_i) = 1$. Then,

$$g_i(\theta_i) = E_{\theta_i}[I_{A_i}(X_i)] = E_{\theta_i}[I_{A_i}(n_i - X_i)]$$

$= E_{1-\theta_i}[I_{A_i}(X_i)] = g_i(1-\theta_i)$, so $g_i(\frac{1}{2} + \theta_i) = g_i(\frac{1}{2} - \theta_i)$. Now, by Lemma 3.2 and Theorem 6.1, $\delta = (\delta_1, \ldots, \delta_k)$ with $\delta_i(x_i) = I_{[n_i/2 - m_i, n_i(x_i)2 + m_i]}$ is a Γ-minimax rule.

A density function $f(x|\theta)$ is said to be a PF (Pólya-Frequency) function if θ is a location parameter and $f(x|\theta)$ is TP. It is known that if X has a PF density $f(x-\theta)$ and $f(x) = f(-x)$, then $|X|$ has a TP density (see Karlin (1968) p. 738).

Hence,

$$\frac{f(x+\theta_2) + f(x-\theta_2)}{f(x+\theta_1) + f(x-\theta_1)} \tag{6.3}$$

is symmetric about 0 and is increasing for $x > 0$ when $\theta_2 > \theta_1 > 0$.

<u>Theorem 6.2</u>. If X_i has a PF density $f_i(x|\theta_i) = f_i(x - \theta_i)$ and $f_i(x) = f_i(-x)$, tnen the assumptions of Theorem 6.1 are satisfied.

Proof: Now A_i defined in (6.1) reduces to

$A_i = \{x + \theta_0 | -t_i \le x \le t_i\}$ by the monotonicity of (6.3)

Then $g_i(\theta_i) = E_{\theta_i}[I_{A_i}(X_i)] = P[-t_i + \theta_0 \le Z_i + \theta_i \le t_i + \theta_0]$

where $Z_i = X_i - \theta_i$. Since Z_i and $-Z_i$ have the same distribution, it follows that so $g_i(\theta_i + \theta_0) = g_i(\theta_0 - \theta_i)$. By Lemma 3.2, the assumptions of Theorem 6.1 are satisfied.

An example where Theorem 6.2 is applied is when θ_i has a double exponential density $f_i(x|\theta_i)=\frac{c_i}{2}e^{-c_i|x_i-\theta_i|}$ for $i=1,2,\ldots,k$. In this case the Γ-minimax rule is $\underset{\sim}{\delta}=(\delta_1,\ldots,\delta_k)$ with

$$\delta_i(x_i)=\begin{cases}1 & \text{if } \dfrac{\lambda_i' L_2}{\lambda_i L_1}\dfrac{e^{-c_i|x_i-\theta_0-\Delta-\epsilon|}+e^{-c_i|x_i-\theta_0+\Delta+\epsilon|}}{e^{-c_i|x_i-\theta_0-\Delta|}+e^{-c_i|x_i-\theta_0+\Delta|}} > 1\\ \\ 0 & \end{cases}$$

7. <u>Bayes rules and minimax rules.</u>

In Section 2, we assumed that partial information about θ is known and is summarized in the class Γ. In this section, we consider two extreme cases, namely, either complete information or no information about θ is known. Then we are interested in the Bayes rules and minimax rules respectively. The problem will be treated under the assumption that θ_0 is unknown and $X_i \sim N(\theta_i,\sigma_i^2)$ for $i=0,1,\ldots,k$. Assume that θ_i has a normal prior distribution with mean α_i and variance β_i^2, then $\theta_i|X_i \sim N(a_i,b_i^2)$ where

$$a_i = \frac{\alpha_i\sigma_i^2+x_i\beta_i^2}{\sigma_i^2+\beta_i^2} \quad\text{and}\quad b_i^2 = \frac{\sigma_i^2\beta_i^2}{\sigma_i^2+\beta_i^2}.$$

With the same loss function as defined in (2.1), it can be easily shown that the Bayes rule is $\underset{\sim}{\delta}^B=(\delta_1^B,\ldots,\delta_k^B)$ where

$$\delta_i^B(\underset{\sim}{x})=\begin{cases}1 & \text{if } L_2P[|\theta_i-\theta_0|>\Delta+\epsilon\,|\,x_i,x_0] > L_1P[|\theta_i-\theta_0|\le\Delta\,|\,x_i,x_0]\\ \\ 0 & \end{cases}$$

$$=\begin{cases}1 & \text{if } \dfrac{L_2[\Phi(-y_i-\Delta_i'-\epsilon_i')+\Phi(y_i-\Delta_i'-\epsilon_i')]}{L_1[\Phi(y_i+\Delta_i')-\Phi(y_i-\Delta_i')]} > 1\\ \\ 0 & \end{cases} \tag{7.1}$$

$$= \begin{cases} 1 & \text{if } |y_i| \le s_i^B \\ 0 & > \end{cases},$$

where $y_i = \frac{a_i - a_0}{b_i'}$, $\Lambda_i' = \frac{\Lambda}{b_i'}$, $\epsilon_i' = \frac{\epsilon}{b_i'}$, $b_i' = \sqrt{b_0^2 + b_i^2}$ and s_i^B is the positive root of the equation (7.1) with inequality replaced by equality. The following theorem determines the minimax rule.

Theorem 7.1. Let $a=(a_1,a_2,\ldots,a_k)$ and $1=(1,1,\ldots,1)$ and let $\delta^M=(\delta_1^M,\ldots,\delta_k^M)$ be the Γ-minimax rule in D' for $\Gamma=\Gamma(a,1-a)$ (see (3.3)). If a_i is chosen such that $q_i(a_i)=1$ for all $i=1,2,\ldots,k$, where

$$q_i(a) = \frac{L_2[\Phi(\frac{t_i(a)+\Lambda+\epsilon}{\sigma_i'}) - \Phi(\frac{-t_i(a)+\Lambda+\epsilon}{\sigma_i'})]}{L_1[\Phi(\frac{-t_i(a)+\Lambda}{\sigma_i'}) + \Phi(\frac{-t_i(a)-\Lambda}{\sigma_i'})]}, \tag{7.2}$$

then δ^M is a minimax rule.

Proof: For $\theta \in G_i$, $R^{(i)}(\theta,\delta_i^M) = L_1 P[|x_i - x_0| \ge t_i(a_i)|\theta_0,\theta_i]$

$$= L_1[\Phi(\frac{-t_i(a_i)-(\theta_i-\theta_0)}{\sigma_i'}) + \Phi(\frac{-t_i(a_i)+(\theta_i-\theta_0)}{\sigma_i'})]$$

$$\le L_1[\Phi(\frac{-t_i(a_i)-\Lambda}{\sigma_i'}) + \Phi(\frac{-t_i(a_i)+\Lambda}{\sigma_i'})].$$

Similarly, for $\theta \in B_i$,

$$R^{(i)}(\theta,\delta_i^M) \le L_2[\Phi(\frac{t_i(a_i)+\Lambda+\epsilon}{\sigma_i'}) - \Phi(\frac{-t_i(a_i)+\Lambda+\epsilon}{\sigma_i'})].$$

If $\theta \notin B_i \cup G_i$, then $R^{(i)}(\theta,\delta_i^M)=0$. Now from (4.4) and (4.6), we get

$$\liminf_{n\to\infty} r^{(i)}(\tau_n, \delta^0_{in})$$

$$\geq L_1 a[\Phi(\frac{-t_i(a_i)+\Delta}{\sigma_i'})+\Phi(\frac{-t_i(a_i)-\Delta}{\sigma_i'})]$$

$$+ L_2(1-a)[\Phi(\frac{t_i(a_i)+\Delta+\varepsilon}{\sigma_i'})-\Phi(\frac{-t_i(a_i)+\Delta+\varepsilon}{\sigma_i'})]$$

$$= L_1[\Phi(\frac{-t_i(a_i)+\Delta}{\sigma_i'})+\Phi(\frac{-t_i(a_i)-\Delta}{\sigma_i'})]$$

$$= L_2[\Phi(\frac{t_i(a_i)+\Delta+\varepsilon}{\sigma_i'})-\Phi(\frac{-t_i(a_i)+\Delta+\varepsilon}{\sigma_i'})]$$

$$\geq \sup_{\theta\in\Theta} R^{(i)}(\theta, \delta^M_i) \text{ for all } i=1,\ldots,k.$$

Then, $$\liminf_{n\to\infty} r(\tau_n, \delta^0_n) \geq \sum_{i=1}^{k} \liminf_{n\to\infty} r^{(i)}(\tau_n, \delta^0_{in})$$

$$\geq \sum_{i=1}^{k} \sup_{\theta\in\Theta} R^{(i)}(\theta, \delta^M_i) \geq \sup_{\theta\in\Theta} R(\theta, \delta^M).$$

It follows that δ^M is a minimax rule.

Let $$\gamma_i(a,x) = \frac{L_2(1-a)}{L_1 a}\,\frac{f_{\sigma_i'}(x)}{g_{\sigma_i'}(x)}, \tag{7.3}$$

then $\gamma_i(a,t_i(a))=1$ by (4.2), which implies that $t_i(a)$ is a continuous function of a by the implicit function theorem. Hence $q_i(a)$ is a continuous function of a, for $0<a<1$. Now, $\lim_{a\to 1} q_i(a)=\lim_{t_i(a)\to\infty} q_i(a) = \infty$ by (7.1) and (7.2). Also, for

$$a_i^0 = \frac{L_2\exp[-\varepsilon(2\Delta+\varepsilon)/2\sigma_i^2]}{L_1+L_2\exp[-\varepsilon(2\Delta+\varepsilon)/2\sigma_i^2]},$$ $t_i(a_i^0)=0$, so $q_i(a_i^0)=0$. Then, by the continuity of q_i there exists an $a_i(a_i^0<a_i<1)$ such that $q_i(a_i)=1$. This shows that a minimax rule always exists.

8. Comparison among Bayes, Γ-minimax and minimax rules

When one faces a decision problem, the choice of the optimal rules depends on the prior information one has. In general, one would use Bayes rules if the prior distribution is known exactly, use Γ-minimax rule for incomplete prior information and use minimax rule if no prior information is available. Hence one is interested in studying the robustness of these rules against the assumption of the prior information. In this section, we compare these rules in terms of the Bayes risk, maximum risk over Γ and the maximum risk over Θ^*. Since the loss function is assumed to be additive, the comparison is made for the first component problem only. In this section, $\underset{\sim}{x}=(x_0,x_1)$, $\underset{\sim}{\theta}=(\theta_0,\theta_1)$ and $d\tau_B(\theta)=d\tau_0(\theta_0)d\tau_1(\theta_1)$, where $\tau_i \sim N(\alpha_i,\beta_i^2)$ for $i=0,1$. Let $\delta_B(\underset{\sim}{x})=I_{[-t_B,t_B]}(a_1-a_0)$ be the Bayes rule wrt τ_B (see Section 7), where $a_i = \dfrac{\alpha_i\sigma_i^2+x_i\beta_i^2}{\sigma_i^2+\beta_i^2}$ for $i=0,1$. Also, let $\delta_G(\underset{\sim}{x})=I_{[-t_G,t_G]}(x_1-x_0)$ be the Γ-minimax rule in D' and $\delta_M(x)=I_{[-t_M,t_M]}(x_1-x_0)$ be the minimax rule. Define $r_1(\delta)=r^{(1)}(\tau_B,\delta)$, $r_2(\delta)=\sup\limits_{\tau\in\Gamma} r^{(1)}(\tau,\delta)$ and $r_3(\delta)=\sup\limits_{\tau\in\Theta^*} r^{(1)}(\tau,\delta)$.

Then, $r_1(\delta_B)=L_1P_{\tau_B}[|a_1-a_0|<t_B, |\theta_1-\theta_0|\geq\Delta]+L_2P_{\tau_B}[|a_1-a_0|\geq t_B, |\theta_1-\theta_0|<\Delta]$.

Let $d = \alpha_1-\alpha_0$, $w_i^2= \dfrac{\beta_i^4}{\sigma_i^2+\beta_i^2}$, $i=0,1$

$u^2=\beta_0^2+\beta_1^2$ and $v^2=w_0^2+w_1^2$. Then, we find that

$$\begin{pmatrix} \dfrac{a_1-a_0-d}{v} \\ \dfrac{\theta_1-\theta_0-d}{u} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \quad \text{where } \rho = \frac{v}{u}.$$

Hence $r_1(\delta_B) =$

$$L_1\{F(-B_1,C;-\rho)+F(-B_1,D;-\rho)+F(B_2,C;\rho)+F(B_2,D;\rho)\}$$

$$+ L_2\{F(B_1,D-A;\rho)+F(B_2,D-A;\rho)+F(B_1,-C-A;\rho)+F(B_2,-C-A;-\rho)\},$$

where $B_1 = \dfrac{t_B-d}{v}$, $B_2 = \dfrac{-t_B-d}{v}$, $A = \dfrac{\varepsilon}{u}$, $C = \dfrac{\Delta-d}{u}$, $D = \dfrac{-\Delta-d}{u}$ and

$$F(x_0,y_0;\rho)=P[Z_1\le x_0, Z_2\le y_0] \text{ with } \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

Similarly, $r_1(\delta_G)=L_1P_{\tau_B}[|X_1-X_0|>t_G, |\theta_1-\theta_0|\le\Delta]+L_2P_{\tau_B}[|X_1-X_0|\le t_G, |\theta_1-\theta_0|\ge\Delta+\varepsilon]$

$$= L_1[F(-G_1,C;-\rho')+F(-G_1,D;-\rho')+F(G_2,C;\rho')+F(G_2,D;\rho')]$$

$$+ L_2[F(G_1,D-A;\rho')+F(G_2,D-A;\rho')+F(G_1,-C-A;-\rho')+F(G_2,-C-A;-\rho')],$$

where $G_1 = \dfrac{t_G-d}{\gamma}$, $G_2 = \dfrac{t_G-d}{\gamma}$, $\gamma^2=\sigma_0^2+\sigma_1^2+\mu^2$ and $\rho' = \dfrac{u}{\gamma}$.

Since δ_G and δ_M have the same form except for the constant t_G and t_M, so if we let $M_1 = \dfrac{t_M-d}{\gamma}$ and $M_2 = \dfrac{-t_M-d}{\gamma}$ and replaces G_1, G_2 by M_1, M_2, respectively, in the above formula, we get $r_1(\delta_M)$. The following lemma is used to compute the maximum risk over Γ.

<u>Lemma 8.1</u>. $r_2(\delta)=L_1\lambda_1(1-\inf\limits_{|\theta_1-\theta_0|\le\Delta} E_\theta[\delta(X)])+L_2\lambda_1' \sup\limits_{|\theta_1-\theta_2|\ge\Delta+\varepsilon} E_\theta[\delta(X)]$.

Proof: $\le$ is trivial. To prove the other inequality, let $\{\theta_n\}$ and $\{\theta_n'\}$ be two sequences such that $\theta_n \in \{\theta \mid |\theta_1-\theta_0|\le\Delta\}$, $\theta_n' \in \{\theta \mid |\theta_1-\theta_0|\ge\Delta+\varepsilon\}$ and $E_{\theta_n}[\delta(X)] \to \inf\limits_{|\theta_1-\theta_0|\le\Delta} E_\theta[\delta(X)]$,

$E_{\theta_n'}[\delta(X)] \to \sup_{|\theta_1-\theta_0|\leq\Lambda} E_\theta[\delta(X)]$. Let $\tau_n\in\Gamma$ be defined by $P_{\tau_n}[\theta=\theta_n]=\lambda_1$ and $P_{\tau_n}[\theta=\theta_n']=\lambda_1'$, then

$$r_2(\delta)=\sup_{\tau\in\Gamma} r^{(1)}(\tau,\delta)\geq\lim_{n\to\infty} r^{(1)}(\tau_n,\delta)=L_1\lambda_1[1-\inf_{|\theta_1-\theta_0|\leq\Lambda} E_\theta[\delta(X)])$$

$$+ L_2\lambda_1' \sup_{|\theta_1-\theta_0|\geq\Lambda+\varepsilon} E_\theta[\delta(X)].$$ This finishes the proof.

From Lemma 8.1 and (4.5), we get $r_2(\delta_G)=L_1\lambda_1[\Phi(\frac{-t_G-\Lambda}{\sigma_1'})+\Phi(\frac{-t_G+\Lambda}{\sigma_1'})]$

$$+ L_2\lambda_1'[\Phi(\frac{t_G-\Lambda-\varepsilon}{\sigma_1'})-\Phi(\frac{-t_G-\Lambda-\varepsilon}{\sigma_1'})].$$

When t_G is replaced by t_M, we get $r_2(\delta_M)=(\lambda_1+\lambda_2)L_1[\Phi(\frac{-t_M+\Lambda}{\sigma_1'})+\Phi(\frac{-t_M-\Lambda}{\sigma_1'})]$.

To find $r_2(\delta_B)$, first note that $a_1-a_0|\theta_1,\theta_0 \sim N(\mu,\zeta^2)$ where

$$\mu=\frac{\beta_1^2\theta_1}{\beta_1^2+\sigma_1^2}-\frac{\beta_0^2\theta_0}{\beta_0^2+\sigma_0^2}+(\frac{\alpha_1\sigma_1^2}{\beta_1^2+\sigma_1^2}-\frac{\alpha_0\sigma_0^2}{\beta_0^2+\sigma_0^2}) \text{ and } \zeta^2=\frac{\beta_1^4\sigma_1^2}{(\beta_1^2+\sigma_1^2)^2}+\frac{\beta_0^4\sigma_0^2}{(\beta_0^2+\sigma_0^2)^2}.$$ Let

$$g_B(\mu)=E_\theta[\delta_B(X)]=P_\mu[-t_B\leq a_1-a_0\leq t_B]=\Phi(\frac{t_B-\mu}{\zeta})-\Phi(\frac{-t_B-\mu}{\zeta}), \tag{8.1}$$

then $g_B(\mu)=g_B(-\mu)$ and $g_B(\mu)$ is decreasing in $|\mu|$.

We consider the following two cases:

(a) if $\frac{\sigma_0^2}{\beta_0^2}\neq\frac{\sigma_1^2}{\beta_1^2}$ then $\frac{\beta_1^2}{\beta_1^2+\sigma_1^2}\neq\frac{\beta_0^2}{\beta_0^2+\sigma_0^2}$. Let $\theta_1=\theta_0\to\pm\infty$ then $|\mu|\to\infty$.

So, $\inf_{|\theta_1-\theta_0|\leq\Lambda} E_\theta[\delta_B(X)]=\lim_{|\mu|\to\infty} g_B(\mu)=0$ and $\sup_{|\theta_1-\theta_0|\geq\Lambda+\varepsilon} E_\theta[\delta_B(X)]=g_B(0)$,

because $\{\theta|\mu=0\}\cap\{\theta||\theta_1-\theta_0|\geq\Lambda+\varepsilon\}\neq\emptyset$. Hence, $r_2(\delta_B)=L_1\lambda_1+L_2\lambda_1'g_B(0)$.

(b) If $\frac{\sigma_0^2}{\beta_0^2}=\frac{\sigma_1^2}{\beta_1^2}=e^2$, then $\mu=\frac{1}{1+e^2}[(\theta_1-\theta_0)+e^2(\alpha_1-\alpha_0)]$. So, when

$|\theta_1-\theta_0|\leq\Lambda$, the maximum value of $|\mu|$ is $\mu_0=\frac{1}{1+e^2}[\Lambda+e^2|\alpha_1-\alpha_0|]$.

(8.2)

When $|\theta_1-\theta_0|\geq\Delta+\varepsilon$, the minimum value of $|\mu|$ is 0 if $e^2|\alpha_1-\alpha_0|\geq\Delta+\varepsilon$; and is

$$\mu_1=\frac{1}{1+e^2}\left[(\Delta+\varepsilon)-e^2|\alpha_1-\alpha_0|\right] \text{ if } e^2|\alpha_1-\alpha_0|\leq\Delta+\varepsilon. \tag{8.3}$$

Hence, we get

$$r_2(\delta_B)=\begin{cases} L_1\lambda_1[1-g_B(\mu_0)]+L_2\lambda_1' g_B(0) & \text{if } e^2|\alpha_1-\alpha_0|\geq\Delta+\varepsilon \\ L_1\lambda_1[1-g_B(\mu_0)]+L_2\lambda_1' g_B(\mu_1) & \text{if } e^2|\alpha_1-\alpha_0|\leq\Delta+\varepsilon, \end{cases}$$

where g_B is defined in (8.1). To find $r_3(.)$, we need the following lemma.

<u>Lemma 8.2.</u> $r_3(\delta)=\max\{L_1(1-\inf_{|\theta_1-\theta_0|\leq\Delta} E_{\underset{\sim}{\theta}}[\delta(\underset{\sim}{X})]),\ L_2\sup_{|\theta_1-\theta_0|\geq\Delta+\varepsilon} E_\theta[\delta(x)]\}$.

Proof: The proof is similar to that of Lemma 8.1.

Now, from Theorem 7.1, $r_3(\delta_M)=L_1[\Phi(\frac{-t_M-\Delta}{\sigma_1'})+\Phi(\frac{-t_M+\Delta}{\sigma_1'})] = L_2[\Phi(\frac{t_M-\Delta-\varepsilon}{\sigma_1'})-\Phi(\frac{-t_M-\Delta-\varepsilon}{\sigma_1'})]$. From (4.5), $r_3(\delta_G)=\max\{L_1[\Phi(\frac{-t_G-\Delta}{\sigma_1'}) + \Phi(\frac{-t_G+\Delta}{\sigma_1'})],\ L_2[\Phi(\frac{t_G-\Delta-\varepsilon}{\sigma_1'})-\Phi(\frac{-t_G-\Delta-\varepsilon}{\sigma_1'})]\}$. We also find that $r_3(\delta_B)=\max\{L_1,L_2 g_B(0)\}$ if $\frac{\sigma_0^2}{\beta_0^2}\neq\frac{\sigma_1^2}{\beta_1^2}$. For $\frac{\sigma_0^2}{\beta_0^2}=\frac{\sigma_1^2}{\beta_1^2}=e^2$,

$$r_3(\delta_B)=\begin{cases} \text{Max}\{L_1(1-g_B(\mu_0)),L_2 g_B(0)\} & \text{if } e^2|\alpha_1-\alpha_0|\geq\Delta+\varepsilon \\ \text{Max}\{L_1(1-g_B(\mu_0),\ L_2 g_B(\mu_1)\} & \text{if } \quad < \end{cases}$$

where μ_0 and μ_1 are defined in (8.2) and (8.3). Thus we have all the formulas needed to compute the tables for comparison.

Table I, II exhibit t_B, t_G, t_M and $r_i(\delta)$ for $\delta = \delta_B, \delta_B$, and δ_M, i=1,2,3. They are arranged in the following manner:

$$\begin{array}{c|ccc} t_B & r_1(\delta_B) & r_2(\delta_B) & r_3(\delta_B) \\ t_G & r_1(\delta_G) & r_2(\delta_G) & r_3(\delta_G) \\ t_M & r_1(\delta_M) & r_2(\delta_M) & r_3(\delta_M) \end{array}$$

The tables are computed with $\frac{\sigma_1^2}{n_1} = \frac{\sigma_0^2}{n_0} = \sigma^2$ and $(\alpha_0, \beta_0^2) = (0,1)$. The selected values of the variables are:

(1) σ^2 is .2 in Table I and is .5 in Table II.

(2) (α_1, β_1^2) is chosen as (0,.5), (0,1), or (0,2).

(3) Δ is chosen as 1. or 1.5.

(4) For Δ=1., ε is chosen as .3 or .8.

For Δ=1.5, ε is chosen as .5 or 1.

(5) For (α_1, β_1^2), Λ, ε, and σ^2 fixed, λ_1 and λ_1' are computed so that $\tau_B \in \Gamma$.

TABLE I. $\rho^2 = .2$

$\Delta=1$	$(\alpha_1,\beta_1^2)=(0,.5)$			$\Delta=1.5$	$(\alpha_1,\beta_1^2)=(0,.5)$		
$\varepsilon=.3$	$\lambda_1=.5858$	$\lambda_1'=.2885$		$\varepsilon=.5$	$\lambda_1=.7793$	$\lambda_1'=.1025$	
1.1500	.1303	.8687	1.0	1.7500	.0524	.8818	1.0
2.0944	.1798	.2828	.8955	3.3731	.0889	.1021	.9850
1.1503	.1512	.3553	.4064	1.7500	.0771	.3054	.3463
$\varepsilon=.8$	$\lambda_1=.5858$	$\lambda_1'=.1416$		$\varepsilon=1.0$	$\lambda_1=.7793$	$\lambda_1'=.0412$	
1.4000	.0508	.7268	1.0	2.0000	.0169	.8206	1.0
2.1098	.0619	.1207	.6879	3.1757	.0264	.0385	.8573
1.4001	.0708	.1917	.2636	2.0000	.0351	.1761	.2146
$\Delta=1$	$(\alpha_1,\beta_1^2)=(0,1)$			$\Delta=1.5$	$(\alpha_1,\beta_1^2)=(0,1)$		
$\varepsilon=.3$	$\lambda_1=.5205$	$\lambda_1'=.3580$		$\varepsilon=.5$	$\lambda_1=.7112$	$\lambda_1'=.1573$	
1.1499	.1337	.3397	.5503	1.7500	.0647	.2104	.5628
1.6494	.1461	.3333	.7097	2.9570	.1066	.1546	.9349
1.1503	.1447	.3570	.4064	1.7500	.0804	.3008	.3463
$\varepsilon=.8$	$\lambda_1=.5205$	$\lambda_1'=.2031$		$\varepsilon=1.0$	$\lambda_1=.7112$	$\lambda_1'=.0771$	
1.4000	.0576	.1597	.4247	2.0000	.0241	.0887	.4372
1.8706	.0619	.1545	.5444	2.8887	.0344	.0663	.7306
1.4001	.0690	.1907	.2636	2.0000	.0367	.1692	.2146
$\Delta=1$	$(\alpha_1,\beta_1^2)=(0,2)$			$\Delta=1.5$	$(\alpha_1,\beta_1^2)=(0,2)$		
$\varepsilon=.3$	$\lambda_1=.4363$	$\lambda_1'=.4529$		$\varepsilon=.5$	$\lambda_1=.6135$	$\lambda_1'=.2482$	
1.1499	.1260	.8724	1.0	1.7500	.0725	.8614	1.0
1.1044	.1350	.3612	.4349	2.4739	.0982	.2298	.7732
1.1503	.1318	.3614	.4064	1.7500	.0810	.2984	.3463
$\varepsilon=.8$	$\lambda_1=.4363$	$\lambda_1'=.2987$		$\varepsilon=1.0$	$\lambda_1=.6135$	$\lambda_1'=.1489$	
1.4000	.0588	.7317	1.0	2.0000	.0304	.7624	1.0
1.5896	.0597	.1871	.3697	2.5663	.0375	.1088	.5418
1.4001	.0648	.1937	.2636	2.0000	.0376	.1636	.2146

TABLE II. $\sigma^2 = .5$

$\Lambda=1$	$(\alpha_1,\beta_1^2)=(0,.5)$			$\Lambda=1.5$	$(\alpha_1,\beta_1^2)=(0,.5)$		
$\varepsilon=.3$	$\lambda_1=.5858$	$\lambda_1'=.2885$		$\varepsilon=.5$	$\lambda_1=.7793$	$\lambda_1'=.1025$	
1.1469	.1968	.8594	1.0	1.7500	.0803	.8815	1.0
3.5136	.2654	.2881	.9866	5.8077	.1022	.1025	.9999
1.1771	.2522	.3886	.4445	1.7509	.1579	.3541	.4015
$\varepsilon=.8$	$\lambda_1=.5858$	$\lambda_1'=.1416$		$\varepsilon=1.0$	$\lambda_1=.7793$	$\lambda_1'=.0412$	
1.3991	.0928	.7249	1.0	2.0000	.0313	.8205	1.0
3.1767	.1131	.1384	.9157	4.9393	.0398	.0412	.9920
1.4117	.1562	.2533	.3482	2.0003	.0979	.2533	.3087
$\Lambda=1$	$(\alpha_1,\beta_1^2)=(0,1)$			$\Lambda=1.5$	$(\alpha_1,\beta_1^2)=(0,1)$		
$\varepsilon=.3$	$\lambda_1=.5205$	$\lambda_1'=.3580$		$\varepsilon=.5$	$\lambda_1=.7112$	$\lambda_1'=.1573$	
1.1441	.2113	.3613	.6601	1.7500	.1066	.2081	.7340
2.4180	.2430	.3516	.8681	4.7675	.1517	.1572	.9972
1.1771	.2412	.3904	.4445	1.7509	.1592	.3487	.4015
$\varepsilon=.8$	$\lambda_1=.5205$	$\lambda_1'=.2031$		$\varepsilon=1.0$	$\lambda_1=.7112$	$\lambda_1'=.0771$	
1.3980	.1121	.1967	.6167	2.0000	.0493	.1008	.6915
2.5834	.1238	.1887	.7833	4.2218	.0659	.0761	.9574
1.4117	.1506	.2520	.3482	2.0003	.0979	.2433	.3087
$\Lambda=1$	$(\alpha_1,\beta_1^2)=(0,2)$			$\Lambda=1.5$	$(\alpha_1,\beta_1^2)=(0,2)$		
$\varepsilon=.3$	$\lambda_1=.4363$	$\lambda_1'=.4529$		$\varepsilon=.5$	$\lambda_1=.6135$	$\lambda_1'=.2482$	
1.1409	.2058	.8343	1.0	1.7499	.1269	.8574	1.0
1.1754	.2212	.3952	.4452	3.5599	.1833	.2456	.9406
1.1771	.2210	.3952	.4445	1.7509	.1563	.3460	.4015
	$\lambda_1=.4363$	$\lambda_1'=.2987$		$\varepsilon=1.0$	$\lambda_1=.6135$	$\lambda_1'=.1489$	
1.3968	.1205	.7177	1.0	2.0000	.0679	.7615	1.0
1.8999	.1230	.2423	.5397	3.4159	.0878	.1391	.8201
1.4117	.1409	.2559	.3482	2.0003	.0967	.2353	.3087

Discussion of the Tables

It is seen from Table I and II that:

1. Minimax rules compare favorably with Γ-minimax rules in terms of $r_2(.)$, and with Bayes rules in terms of the risk $_1(.)$.
2. The Bayes risk of the Γ-minimax rules is only a little more than that of Bayes rules.
3. When $(\alpha_1,\beta_1^2)=(0,1)$, the performance of Bayes rules is close to Γ-minimax rule in terms of $r_2(.)$ and close to that of the minimax rule in terms of $r_3(.)$. If $(\alpha_1,\beta_1^2)\neq(0,1)$, Bayes rules show some large increase of risks $\gamma_2(.)$ and $\gamma_3(.)$ when compared with Γ-minimax rules and minimax rules, respectively. To illustrate the use of the tables, let us look at the following example:

Example 8.1. Type Π_0 (control) machines produce part $P(p)$ where p is the diameter of P, and $p|\Pi_0 \sim N(\theta_0,1)$. Type Π_1,Π_2, and Π_3 machines produce part $Q_i(q_i)$, and $q_i|\Pi_i \sim N(\theta_i,1)$ for $i=1,2,3$. Let us assume that when $|\theta_i-\theta_0| \leq 1.5$, part P and part Q_i can be matched, and when $|\theta_i-\theta_0| \geq 2.5$ they cannot be matched. Assume that the partial prior information Γ is as follows:

$P[|\theta_1-\theta_0| \leq 1.5]=.78$ $\qquad$ $P[|\theta_1-\theta_0| \geq 2.5]=.04$

$P[|\theta_2-\theta_0| \leq 1.5]=.71$ $\qquad$ $P[|\theta_2-\theta_0| \geq 2.5]=.08$

$P[|\theta_3-\theta_0| \leq 1.5]=.61$ $\qquad$ $P[|\theta_3-\theta_0| \geq 2.5]=.15$

Now, there are machines a_0, a_1, a_2, a_3 for sale where $a_i \in \Pi_i$ for $i=0,1,2,3$. Suppose we can take 5 samples from each machine and let $\bar{X}_i$ be the mean diameter of the samples from machine a_i $(i=0,1,2,3)$. Since $\Delta=1.5$, $\varepsilon = 1.0$, from Table I, the Γ-minimax rule is:

a_1 is good for a_0 iff $|\bar{X}_1-\bar{X}_0| \leq 3.1757$

a_2 is good for a_0 iff $|\bar{X}_2-\bar{X}_0| \leq 2.8887$

a_3 is good for a_0 iff $|\bar{X}_3-\bar{X}_0| \leq 2.5663$

If we feel the claim regarding the partial prior may not be correct and we would rather assume that there is no prior information, then we might use the following minimax rule: a_i is good for a_0 iff $|\bar{X}_i-\bar{X}_0| \leq 2.0$ for i=1,2,3.

If from some other source, we know that $\theta_0 \sim N(0,1)$, $\theta_1 \sim N(0,5)$, $\theta_2 \sim N(0,1)$ and $\theta_3 \sim N(0,2)$. Then, we might use the Bayes rules, from Table I we get

a_1 is good for a_0 if $|\frac{5}{7}\bar{X}_1- \frac{5}{6}\bar{X}_0| \leq 2.0$

a_2 is good for a_0 if $|\bar{X}_2-\bar{X}_0| \leq 2.4$

a_3 is good for a_0 if $|\frac{10}{11}\bar{X}_3- \frac{5}{6}\bar{X}| \leq 2.0$

If we are not sure about the definiteness of any prior information, we may then use the rule which is most robust to the assumption of the prior distribution. So from Table I, we may use Γ-minimax rule for a_1, use Bayes rule for a_2 and use minimax rule for a_3.

REFERENCES

[1] Bechhofer, R. E., and Turnbull, B. N. (1974). A (k+1) - decision single-stage selection procedure for comparing k normal means with a fixed known standard; the case of common unknown variance. Tech. Report No. 242. Dept. of Operations Res., Cornell University, Ithaca, New York.

[2] Dunnett, C. W. (1955). A multiple comparison procedure for comparing several treatments with a control. J. Amer. Statist. Assoc., 50, 1096-1121.

[3] Ferguson, T. S. (1967). Mathematical Statistics - A decision-theoretic approach. Academic Press, N.Y.

[4] Gupta, S. S., and Sobel, M. (1958). On selecting a subset which contains all populations better than a control. Ann. Math. Statistics, 29, 235-244.

[5] Gupta, S. S., and Kim, W. C. (1980). Γ-minimax and minimax rules for comparison of treatments with a control. To appear in the Proceedings of the International Conference in Statistics (Ed. K. Matusita), North Holland Publishing Co., Amsterdam, The Netherlands.

[6] Gupta, S. S., and Singh, A. K. (1979). On selection rules for treatments versus control problems. Proceedings of the 42nd Session of the International Statistical Institute. December 1979, Manila, Philippines; contributed papers, pp. 229-232.

[7] Huang, W. T. (1975). Bayes approach to a problem of partitioning k normal populations. Bull. Inst. Math. Acad. Sinica, 3, 87-97.

[8] Karlin, S. (1968). Total Positivity. Stanford University Press, Stanford, California.

[9] Kim, W. C. (1979). On Bayes and Gamma-minimax subset selection rules. Ph.D. thesis, Dept. of Statistics. Purdue University, West Lafayette, Indiana.

[10] Lehmann, E. L. (1959). Testing Statistical Hypotheses. John Wiley, New York.

[11] Lehmann, E. L. (1961). Some Model I problems of selection. Ann. Math. Statist., 32, 990-1012.

[12] Miescke, K. J. (1979). Γ-minimax selection procedures in simultaneous testing problems. Mimeo. Ser. No. 79-1, Dept. of Statistics, Purdue University, West Lafayette, Indiana. To appear in the Ann. of Statist.

[13] Paulson, E. (1952). On the comparison of several experimental categories with a control. Ann. Math. Statist., 20, 95-98.

[14] Randles, R. H., and Hollander, M. (1971). Γ-minimax selection procedures in treatment versus control problems. Ann. Math. Statist. 42, 330-341.

[15] Singh, A. K. (1977). On slippage test and multiple decision procedures. Ph.D. Thesis, Dept. of Statistics, Purdue University, West Lafayette, Indiana.

[16] Tong, Y. L. (1969). On partitioning a set of normal populations by their locations with respect to a control. Ann. Math. Statist., 40, 1300-1324.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Mimeograph Series #80-9	2. GOVT ACCESSION NO. AD 096 099	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On Γ-Minimax, Minimax, and Bayes Procedures for Selecting Populations Close to a Control.		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER Mimeo. Series #80-9
7. AUTHOR(s) S. S. Gupta and Ping Hsiao		8. CONTRACT OR GRANT NUMBER(s) ONR N00014-75-C-0455
9. PERFORMING ORGANIZATION NAME AND ADDRESS Purdue University Department of Statistics, W. Lafayette, IN 47907		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Washington, DC		12. REPORT DATE April 1980
		13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Γ-minimax rules, partial prior information, minimax procedures, Bayes procedures, subset selection, robustness.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\Pi_0, \Pi_1, \ldots, \Pi_k$ be $(k+1)$ normally distributed populations and let Π_0 be a control population. Our goal is to select those populations which are sufficiently close to the control in terms of the (unknown) means of the populations. A zero-one type loss function is defined. Γ-minimax rules, Bayes rules and minimax rules are derived for this problem and compared. Some optimal properties of Γ-minimax rules are shown; also Γ-minimax rules are

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) derived for distributions other than the normal.

ATE

LMED

-8